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A note on the asymptotic formula for solutions of the linealized Gel'fand problem[†1]

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Abstract

The purpose of this note is to present another approach to the proof of the asymptotic formula for the solutions of the linealized Gel'fand problem in two space dimensions. The formula is a key lemma in our recent paper concerning the asymptotic non-degeneracy for the Gel'fand problem in two space dimensions.

1 Introduction

In the recent paper [5], we are concerned with the asymptotic behavior of solutions for the the Gel'fand problem as the non-negative parameters $\lambda \longrightarrow 0$:

$$-\Delta u = \lambda e^{u} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$.

More precisely, let $\{\lambda_n\}_{n\in\mathbf{N}}$ be a sequence satisfying $\lambda_n \downarrow 0$ and $u = u_n(x)$ be a solution to (1.1) for $\lambda = \lambda_n$. The possible asymptotic behaviors of u_n as $n \longrightarrow \infty$ are well-studied by Nagasaki-Suzuki[8] in terms of $\Sigma_n = \lambda_n \int_\Omega e^{u_n}$. They established that $\{\Sigma_n\}$ accumulates to Σ_∞ which is either 0, $8\pi m$ for some positive integer m, or $+\infty$. We are concerned with the cases $\Sigma_\infty = 8\pi m$, where the (sub-)sequence of solutions $\{u_n\}$ is known to blow-up at m-points, that is, there is a blow-up set $\mathscr{S} = \{\kappa_1, \ldots, \kappa_m\} \subset \Omega$ of distinct m-points such that $\|u_n\|_{L^\infty(\omega)} = O(1)$ for every $\omega \subset\subset \overline{\Omega}\backslash\mathscr{S}$ and $\{u_n(x)\}$ have a limit for $x \in \overline{\Omega}\backslash\mathscr{S}$ while $u_n|_{\mathscr{S}} \longrightarrow +\infty$. In this case, the limiting function u_∞ has the form

$$u_\infty(x) = 8\pi \sum_{j=1}^{m} G(x, \kappa_j), \tag{1.2}$$

where $G(x,y)$ is the Green function of $-\Delta$ under the Dirichlet condition, that is,

$$-\Delta G(\cdot, y) = \delta_y \quad \text{in } \Omega, \qquad G(\cdot, y) = 0 \quad \text{on } \partial\Omega.$$

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Furthermore, the blow-up points $\kappa_i (i = 1, \cdots, m)$ satisfy the relations

$$\nabla \left[K(x, \kappa_j) + \sum_{i \neq j} G(x, \kappa_i) \right] \Bigg|_{x=\kappa_j} = 0 \quad (1 \leqq j \leqq m), \tag{1.3}$$

where $K(x, y) = G(x, y) - \frac{1}{2\pi} \log |x - y|^{-1}$.

Now we introduce the function

$$H^m(x_1, \ldots, x_m) = \frac{1}{2} \sum_{i=1}^{m} R(x_i) + \frac{1}{2} \sum_{\substack{1 \leq i,j \leq m \\ i \neq j}} G(x_i, x_j),$$

which we call *the Hamiltonian.* Here $R(x) = K(x, x)$ is the Robin function of Ω. Then the relation (1.3) means that $\mathscr{S} \in \Omega^m$ is *a critical point* of the function H^m of $2m$-variables. Therefore we may say that the limit function of $\{u_n\}$ blows up at the critical point of the Hamiltonian H^m. The main result in [5] shows a deeper link between H^m and $\{u_n\}$.

Let us introduce the functional

$$F_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega e^u dx$$

over $H_0^1(\Omega)$, which leads (1.1) as an Euler-Lagrange equation. Then we get the following:

Theorem 1.1 ([5, Theorem 1.2]). *Suppose $\mathscr{S}$ is a non-degenerate critical point of H^m. Then u_n is a non-degenerate critical point of F_{λ_n} for n large enough.*

This kind of result is sometimes called *the asymptotic nondegeneracy* of u_n and it has been already established by Gladiali and Grossi [3] for the case $m = 1$. Similarly to [3], we proved Theorem 1.1 arguing by contradiction. For this purpose we assumed the existence of a sequence $\{v_n\}$ of non-degenerate critical point of F_{λ_n} as $n \longrightarrow \infty$. Using a standard argument, we see that v_n is a non-trivial solution of the linearized problem of (1.1):

$$-\Delta v = \lambda_n e^{u_n} v \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

Without loss of generality we are able to assume

$$\|v_n\|_{L^\infty(\Omega)} \equiv 1. \tag{1.5}$$

Then we got a contradiction $\|v_n\|_{L^\infty(\Omega)} \longrightarrow 0$ after several calculations.

Take a sufficiently small positive number $\bar{R} > 0$ and a sequence $\{x_{j,n}\}$ for each $\kappa_j \in \mathscr{S}$ satisfying

$$x_{j,n} \longrightarrow \kappa_j, \quad u_n(x_{j,n}) = \max_{B_{\bar{R}}(x_{j,n})} u_n(x) \longrightarrow \infty$$

as $n \longrightarrow \infty$, see [7]. Then we re-scale u_n and v_n around $x_{j,n}$ as follows:

$$\begin{aligned} \widetilde{u}_{j,n}(\widetilde{x}) &= u_n(\delta_{j,n}\widetilde{x} + x_{j,n}) - u_n(x_{j,n}) \quad \text{in } B_{\frac{\bar{R}}{\delta_{j,n}}}(0) \\ \widetilde{v}_{j,n}(\widetilde{x}) &= v_n(\delta_{j,n}\widetilde{x} + x_{j,n}) \quad \text{in } B_{\frac{\bar{R}}{\delta_{j,n}}}(0) \end{aligned} \tag{1.6}$$

where the scaling parameter $\delta_{j,n}$ is chosen to satisfy $\lambda_n e^{u_n(x_{j,n})}\delta_{j,n}^2 = 1$.

Then $\widetilde{u}_{j,n}$ and $\widetilde{v}_{j,n}$ satisfy

$$\begin{cases} -\Delta \widetilde{u}_{j,n} = e^{\widetilde{u}_{j,n}}, & \text{in } B_{\frac{\bar{R}}{\delta_{j,n}}}(0) \\ \widetilde{u}_{j,n} \leqq \widetilde{u}_{j,n}(0) = 0, & \text{in } B_{\frac{\bar{R}}{\delta_{j,n}}}(0) \end{cases}$$

$$\begin{cases} -\Delta \widetilde{v}_{j,n} = e^{\widetilde{u}_{j,n}}\widetilde{v}_{j,n}, & \text{in } B_{\frac{\bar{R}}{\delta_{j,n}}}(0) \\ \|\widetilde{v}_{j,n}\|_{L^\infty\left(B_{\frac{\bar{R}}{\delta_{j,n}}}(0)\right)} \leqq 1. & \end{cases}$$

Using standard arguments ([3]) based on the classification results in [2, 1], we are able to get $\boldsymbol{a}_j = (a_{j,1}, a_{j,2}) \in \mathbf{R}^2$, $b_j \in \mathbf{R}$ for each j, and subsequences of $\widetilde{u}_{j,n}$ and $\widetilde{v}_{j,n}$ (denoted by the same symbol) satisfying

$$\begin{aligned} \widetilde{u}_{j,n} &\longrightarrow U(\widetilde{x}) = \log \frac{1}{\left(1 + \frac{|\widetilde{x}|^2}{8}\right)^2}, \\ \widetilde{v}_{j,n} &\longrightarrow \frac{\boldsymbol{a}_j \cdot \widetilde{x}}{8 + |\widetilde{x}|^2} + b_j \frac{8 - |\widetilde{x}|^2}{8 + |\widetilde{x}|^2} = \boldsymbol{a}_j \cdot \nabla \left(-\frac{1}{4}U\right) + \frac{b_j}{2}\left\{\widetilde{\boldsymbol{x}} \cdot \nabla U + 2\right\}, \end{aligned} \tag{1.7}$$

locally uniformly in $\mathbf{R}^2$. We note that the above convergence also holds locally in C^2 from the elliptic regularity theory. During the proof of Theorem 1.1, we proved $\boldsymbol{a}_j = 0$ and $b_j = 0$ for every j.

Under the above preparations, we get the following asymptotic formula which is one of key lemmas in our argument:

Lemma 1.2 ([5, Lemma 2.1], cf. [3, (3.14)]). *There exist $C_j > 0$ ($j = 1, \cdots, m$) and subsequence of v_n satisfying*

$$\frac{v_n}{\lambda_n^{\frac{1}{2}}} \longrightarrow 2\pi \sum_{j=1}^{m} C_j \boldsymbol{a}_j \cdot \nabla_y G(x, \kappa_j) \quad \text{in } C^1\left(\overline{\Omega} \setminus \cup_{j=1}^m B_{2\bar{R}}(\kappa_j)\right). \tag{1.8}$$

The purpose of this note is to give another proof of this, which we discuss in the next section.

We end this section with reviewing the rest of the proof to Theorem 1.1. The rest of the argument is divided into 2 parts. The first one is to show the following fact:

Lemma 1.3 ([5, Lemma 2.2]). *If $\mathscr{S}$ is a non-degenerate critical point of H^m, we have $\boldsymbol{a}_j = 0$ for every $j = 1, \dots, m$ in Lemma 1.2.*

This lemma is obtained by the asymptotic formula (1.8) and the newly obtained Rellich-Pohozaev type identity concerning the Green's function:

Proposition 1.4 ([5, Proposition 2.3]). *Take $z_k \in \Omega$ $(k = 1,2,3)$, $R > 0$ such that $B_R(z_1) \subset\subset \Omega$ and $z_2, z_3 \notin \overline{B_R(z_1)}$. Set*

$$\begin{aligned} &I_{ij}(z_1, z_2, z_3) \\ &:= \int_{\partial B_R(z_1)} \left\{ \frac{\partial}{\partial \nu_x} G_{x_i}(x, z_2) G_{y_j}(x, z_3) - G_{x_i}(x, z_2) \frac{\partial}{\partial \nu_x} G_{y_j}(x, z_3) \right\} d\sigma_x \end{aligned}$$

for $i, j = 1, 2$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then, it holds that

$$I_{ij}(z_1, z_2, z_3) = \begin{cases} 0 & (z_1 \neq z_2,\ z_1 \neq z_3) \\ \frac{1}{2} R_{x_i x_j}(z_1) & (z_1 = z_2 = z_3) \\ G_{x_i y_j}(z_1, z_3) & (z_1 = z_2,\ z_1 \neq z_3) \\ G_{x_i x_j}(z_1, z_2) & (z_1 \neq z_2,\ z_1 = z_3). \end{cases} \tag{1.9}$$

We note that the definition of I_{ij} is independent of R. We also note that the second case of (1.9) is a *localized* version of the known identity

$$-\int_{\partial\Omega} G_{x_i}(x, y) \frac{\partial}{\partial \nu_x} G_{y_j}(x, y) d\sigma_x = \frac{1}{2} R_{x_i x_j}(y),$$

(see [3, Lemma 7] for example), while the other cases describe the correlation of the singularities of the Green's function between $\{z_1, z_2, z_3\}$. Here we see the sketch of the proof of Lemma 1.3 assuming Lemma 1.2 and Proposition 1.4.

Sketch of the proof of Lemma 1.3. Fix $\kappa_j \in \mathscr{S}$ and $R > 2\bar{R} > 0$ satisfying $B_R(\kappa_j) \subset\subset \Omega$ and $\overline{B_R(\kappa_j)} \cap \mathscr{S} = \kappa_j$. Differentiating equation (1.1) with respect to x_i, we get the identity

$$-\Delta u_{x_i} = \lambda e^u u_{x_i}, \tag{1.10}$$

which means that u_{x_i} ($i = 1, 2$) is a solution to (1.4) except for the boundary condition. Therefore, using Green's identity we have

$$\int_{\partial B_R(\kappa_j)} \left(\frac{\partial}{\partial \nu}(u_n)_{x_i} v_n - (u_n)_{x_i} \frac{\partial}{\partial \nu} v_n \right) d\sigma = \int_{B_R(\kappa_j)} \left(\Delta (u_n)_{x_i} v_n - (u_n)_{x_i} \Delta v_n \right) dx$$
$$= 0.$$

From the result in [8] mentioned above, we have

$$(u_n)_{x_i} \longrightarrow 8\pi \sum_{j=1}^{m} G_{x_i}(x, \kappa_j) \quad \text{in } C^1(\overline{\omega})$$

for every $\omega \subset\subset \overline{\Omega} \backslash \mathscr{S}$. Then the relation (1.8) implies

$$0 = \int_{\partial B_R(\kappa_j)} \left\{ \frac{\partial}{\partial \nu}(u_n)_{x_i} \cdot \frac{v_n}{\lambda_n^{\frac{1}{2}}} - (u_n)_{x_i} \frac{\partial}{\partial \nu} \left(\frac{v_n}{\lambda_n^{\frac{1}{2}}} \right) \right\} d\sigma$$
$$\longrightarrow 16\pi^2 \sum_{\substack{1 \le l \le m \\ i' = 1,2}} \left(\sum_{1 \le k \le m} I_{ii'}(\kappa_j, \kappa_k, \kappa_l) \right) C_l a_{l,i'}.$$

Here we use Proposition 1.4 and get

$$\sum_{1 \le k \le m} I_{ii'}(\kappa_j, \kappa_k, \kappa_l) = H^m_{x_{j,i} x_{l,i'}}(x_1, \cdots, x_m) \Big|_{(x_1, \cdots, x_m) = (\kappa_1, \cdots, \kappa_m)},$$

which implies

$$0 = 16\pi^2 \operatorname{Hess}(H^m)|_{(x_1, \cdots, x_m) = (\kappa_1, \cdots, \kappa_m)} \cdot {}^t(C_1 \boldsymbol{a}_1, \cdots, C_m \boldsymbol{a}_m)$$

where Hess (H^m) denotes the Hessian of H^m. Since $(\kappa_1, \cdots, \kappa_m)$ is a non-degenerate critical point of H^m,this Hess(H^m) is invertible. Then we conclude $\boldsymbol{a}_j = 0$ for every $j = 1, \cdots, m$ by $C_j > 0$. $\square$

The final step to get Theorem 1.1 is to show $b_j = 0$ for every j and consequently we show the uniform convergence $v_n \longrightarrow 0$ in Ω, which contradicts $\|v_n\|_{L^\infty(\Omega)} \equiv 1$.

2 More about the asymptotic formula

We recall most parts of the proof of Lemma 1.2 from Section 4 of [5].

Theorem 2.1 ([6], see [5, Theorem 4.1]). *For every fixed $0 < R \ll 1$, there exists a constant C independent of j and $n \gg 1$ such that*

$$\left| u_n(x) - \log \frac{e^{u_n(x_{j,n})}}{\left(1 + \frac{\lambda_n}{8} e^{u_n(x_{j,n})} |x - x_{j,n}|^2\right)^2} \right| \leq C \quad \forall x \in B_R(x_{j,n}).$$

Corollary 2.2 ([5, Corollary 4.2]). *For fixed R, there exists a constant C satisfying*

$$\left| \widetilde{u}_{j,n}(\widetilde{x}) - \log \frac{1}{\left(1 + \frac{1}{8}|\widetilde{x}|^2\right)^2} \right| \leq C \quad \forall \widetilde{x} \in B_{\frac{R}{\delta_{j,n}}}(0)$$

for every j.

Corollary 2.3 ([5, Corollary 4.3]). *For each j there exists a constant $C_j > 0$ and a subsequence of $\delta_{j,n}$ satisfying*

$$\delta_{j,n} = C_j \lambda_n^{\frac{1}{2}} + o\left(\lambda_n^{\frac{1}{2}}\right) \quad \text{as } n \longrightarrow \infty.$$

Here we take a cut-off function $\xi \in C_0^\infty([0, \infty))$ satisfying $\operatorname{supp} \xi \Subset [0, 1)$[†3] and

$$\xi \equiv \begin{cases} 1, & (0 \leq r \leq 1/2) \\ 0, & (1 \leq r) \end{cases}, \qquad 0 \leq \xi \leq 1.$$

Then it follows that

$$\begin{aligned} v_n(x) &= \int_\Omega G(x, y) \lambda_n e^{u_n(y)} v_n(y) dy \\ &= \sum_{j=1}^m \int_\Omega G(x, y) \lambda_n e^{u_n(y)} v_n(y) \xi \left(\frac{|y - x_{j,n}|}{\bar{R}} \right) dy \\ &\quad + \int_\Omega G(x, y) \lambda_n e^{u_n(y)} v_n(y) \left\{ 1 - \sum_{j=1}^m \xi \left(\frac{|y - x_{j,n}|}{\bar{R}} \right) \right\} dy \\ &=: \sum_{j=1}^m \psi_{j,n} + \psi_{0,n}. \end{aligned}$$

Recall that outside from $\kappa_1, .., \kappa_m$ we have that u_n is bounded and then we derive $\|\psi_{0,n}\|_{L^\infty(\Omega)} = O(\lambda_n)$ and hence

$$\frac{\psi_{0,n}}{\lambda_n^{\frac{1}{2}}} = O\left(\lambda_n^{\frac{1}{2}}\right) = o(1) \quad \text{uniformly in } \overline{\Omega}. \tag{2.1}$$

It is the next proposition we give another proof in this note.

[†3]We note that this condition is not assumed originally in [5], though it seems necessary here. Obviously the rest of [5] holds if we choose this ξ.

Proposition 2.4 ([5, Proposition 4.4]). *For each j,*

$$\psi_{j,n}(x) = G(x, x_{j,n})\gamma_{j,n} + 2\pi \boldsymbol{a}_j \cdot \nabla_y G(x, x_{j,n})\delta_{j,n} + o(\delta_{j,n})$$

uniformly for all $x \in \overline{\Omega} \backslash B_{\bar{R}}(x_{j,n})$, where

$$\gamma_{j,n} = \int_{\Omega} \lambda_n e^{u_n(y)} v_n(y) \xi \left(\frac{|y - x_{j,n}|}{\bar{R}} \right) dy.$$

In [5] we proved Proposition 2.4 by the argument used in the proof of [4, Proposition 6.4]. In Section 3 we derive it similar argument to [3, Lemma 6 (p.1345)] instead. We note that the proof in this note is far complicated than we chose in [5]. The author think, however, it worth publishing because it may be used in other problems, where the similar argument used in [5] does not applicable.

Here we proceed the sketch of the proof of Lemma 1.2 assuming Proposition 2.4.

Since $B_{2\bar{R}}(\kappa_j) \supset B_{\bar{R}}(x_{j,n})$ for every j and $n \gg 1$, Proposition 2.4 and Corollary 2.3 imply the following *pre*-asymptotic formula:

$$v_n(x) = \sum_{j=1}^{m} \gamma_{j,n} G(x, x_{j,n}) + 2\pi \lambda_n^{\frac{1}{2}} \sum_{j=1}^{m} C_j \boldsymbol{a}_j \cdot \nabla_y G(x, x_{j,n}) + o\left(\lambda_n^{\frac{1}{2}}\right) \tag{2.2}$$

uniformly in $x \in \overline{\Omega} \backslash \cup_{j=1}^{m} B_{2\bar{R}}(\kappa_j)$ and consequently in $C^1\left(\overline{\Omega} \backslash \cup_{j=1}^{m} B_{2\bar{R}}(\kappa_j)\right)$ from the elliptic regularity theory.

To get the finer asymptotic formula (Lemma 1.2) we need to get

$$\gamma_{j,n} = o\left(\lambda_n^{\frac{1}{2}}\right) \tag{2.3}$$

for some subsequence.

To this purpose we suppose that (2.3) does not hold. Then there exists j satisfying

$$\limsup_{n \longrightarrow \infty} \frac{\lambda_n^{\frac{1}{2}}}{|\gamma_{j,n}|} < \infty.$$

Without loss of generality we may assume

$$r_j := \lim_{n \longrightarrow \infty} \frac{\gamma_{j,n}}{\gamma_{1,n}}, \quad c := \lim_{n \longrightarrow \infty} \frac{\lambda_n^{\frac{1}{2}}}{\gamma_{1,n}}$$

and

$$1 = r_1 \geq r_2 \geq \cdots \geq r_m \geq -1$$

for some subsequence. Then we get

$$\frac{v_n(x)}{\gamma_{1,n}} \longrightarrow \sum_{j=1}^{m} r_j G(x, \kappa_j) + 2\pi c \sum_{j=1}^{m} C_j \boldsymbol{a}_j \cdot \nabla_y G(x, \kappa_j) \tag{2.4}$$

uniformly in $x \in \Omega \setminus \cup_{j=1}^{m} B_{2\bar{R}}(\kappa_j)$.

We take $R > 2\bar{R}$ satisfying

$$B_R(\kappa_j) \subset\subset \Omega, \quad B_R(\kappa_j) \cap B_R(\kappa_k) = \emptyset \quad (j \neq k)$$

and set

$$\bar{u}_n := (x - p) \cdot \nabla u_n + 2.$$

This $\bar{u}_n$ also satisfies (1.4) except for the boundary condition, where $p \in \mathbf{R}^2$ is arbitrary. Taking $p = x_{1,n}$ and using Green's formula again, we are able to get $r_1 = 0$ as the limit of the identity

$$0 = \int_{\partial B_R(x_{1,n})} \left\{ \frac{\partial}{\partial \nu} \bar{u}_n \cdot \frac{v_n}{\gamma_{1,n}} - \bar{u}_n \frac{\partial}{\partial \nu} \left(\frac{v_n}{\gamma_{1,n}} \right) \right\} d\sigma.$$

This contradicts $r_1 = 1$ and we obtain the claim.

3 Another proof of Proposition 2.4

For simplicity, we shall omit j in several characters, e.g., ψ_n as $\psi_{j,n}$, $\widetilde{u}_n$ as $\widetilde{u}_{j,n}, \cdots$. Without loss of generality, furthermore, we may assume $\boldsymbol{a}_j = (a, 0)$ for some $a > 0$ and $\kappa_j = 0$.

From Corollary 2.2 we may assume

$$\left| e^{\widetilde{u}_n} \widetilde{v}_n \xi \left(\frac{|\delta_n \widetilde{y}|}{\bar{R}} \right) \right| \leq \frac{C}{\left(1 + \frac{|\widetilde{y}|^2}{8}\right)^2} \leq \frac{C'}{(1 + |\widetilde{y}_1| + |\widetilde{y}_2|)^4} \quad \text{in } \mathbf{R}^2 \tag{3.1}$$

for another constant $C' > 0$. Here we prepare two lemmas similar to [3, Lemma 6].

Lemma 3.1. *Suppose $f(x) \in C^1(\mathbf{R}^2)$ satisfies*

$$|f(x)| \leq \frac{C}{(1 + |x_1| + |x_2|)^p} \quad \textit{for every } x \in \mathbf{R}^2 \tag{3.2}$$

for some $p > 1$. Then for every $a > 0$, the function

$$w(x_1, x_2) := -\int_{\frac{x_1}{a}}^{\infty} f(at, x_2)dt \tag{3.3}$$

satisfies

$$a\frac{\partial}{\partial x_1}w = f. \tag{3.4}$$

Moreover if $f(x)$ further satisfies

$$\int_{-\infty}^{\infty} f(x_1, x_2)dx_1 = 0 \quad \textit{for every } x_2 \in \mathbf{R}, \tag{3.5}$$

there exist $C' > 0$ satisfying

$$|w(x_1, x_2)| \le \frac{C'}{(1 + |x_1| + |x_2|)^{p-1}} \quad \textit{for every } x_2 \in \mathbf{R}^2. \tag{3.6}$$

Proof. The condition $p > 1$ is necessary to define w and (3.4) is obtained by differentiating (3.3). We may only show (3.6) here.

When $x_1 > 0$, we have

$$\begin{aligned}|w(x_1, x_2)| &\le \int_{\frac{x_1}{a}}^{\infty} |f(at, x_2)|dt \le \int_{\frac{x_1}{a}}^{\infty} \frac{C}{(1 + at + |x_2|)^p}dt \\ &= \frac{C}{(p-1)a} \cdot \frac{1}{(1 + x_1 + |x_2|)^{p-1}} =: \frac{C'}{(1 + |x_1| + |x_2|)^{p-1}}.\end{aligned}$$

When $x_1 < 0$ on the contrary, it hold that

$$w(x_1, x_2) = -\int_{\frac{x_1}{a}}^{\infty} f(at, x_2)dt = \int_{-\infty}^{\frac{x_1}{a}} f(at, x_2)dt$$

from the condition (3.5). Therefore we get

$$\begin{aligned}|w(x_1, x_2)| &\le \int_{-\infty}^{\frac{x_1}{a}} |f(at, x_2)|dt \le \int_{-\infty}^{\frac{x_1}{a}} \frac{C}{(1 - at + |x_2|)^p}dt \\ &= \frac{C}{(p-1)a} \cdot \frac{1}{(1 - x_1 + |x_2|)^{p-1}} = \frac{C'}{(1 + |x_1| + |x_2|)^{p-1}}.\end{aligned}$$

□

Remark 3.2. It is necessary that ω is integrable in $\mathbf{R}^2$ in the proof of Proposition 2.4. Therefore we need $p > 3$.

Lemma 3.3. *Suppose $f(x) \in C^2(\mathbf{R}^2 \backslash \{0\})$ and*

$$|f(x)| \leq \frac{C}{(1+|x|)^p} \quad \text{for every } x \in \mathbf{R}^2 \backslash \{0\} \tag{3.7}$$

for some $p > 2$. Then the function

$$\zeta(x) = \zeta(r\cos\theta, r\sin\theta) := \log\left\{-\frac{1}{r^2}\int_r^\infty tf(t\cos\theta, t\sin\theta)dt\right\} \tag{3.8}$$

for $r > 0$ and $\theta \in [0, 2\pi)$ satisfies

$$\operatorname{div}(xe^\zeta) = f \quad \text{for every } x \in \mathbf{R}^2 \backslash \{0\}. \tag{3.9}$$

Moreover for every $R > 0$ there exists a constant $C' = C'(C, R)$ satisfying

$$|x|e^\zeta \leq \frac{C'}{(1+|x|)^{p-1}} \quad \text{for every } x \in \mathbf{R}^2 \backslash B_R(0). \tag{3.10}$$

Here we further suppose

$$\lim_{r \longrightarrow 0} \int_r^\infty tf(t\cos\theta, t\sin\theta)dt = 0 \quad \text{for every } \theta \in [0, 2\pi). \tag{3.11}$$

Then we have $|x|e^\zeta \in L^1_{\mathrm{loc}}(\mathbf{R}^2)$, which implies $|x|e^\zeta \in L^1(\mathbf{R}^2)$ if $p > 3$ from (3.10). Moreover we have

$$e^\zeta \leq \frac{C}{2} \quad \text{in } \mathbf{R}^2 \backslash \{0\}. \tag{3.12}$$

and

$$\operatorname{div}\left(xe^\zeta\right) = f \quad \text{in } \mathscr{D}'(\mathbf{R}^2). \tag{3.13}$$

To prove Lemma 3.3, it is convenient to prepare the following:

Proposition 3.4. *Suppose f and $g \in C^2(\mathbf{R}^2 \backslash \{0\})$ satisfy*

$$\operatorname{div}(xg) = f \quad \text{in } \mathscr{D}'(\mathbf{R}^2 \backslash \{0\}). \tag{3.14}$$

Moreover suppose

$$f \in L^1_{\mathrm{loc}}\left(\mathbf{R}^2\right) \tag{3.15}$$

and

$$\lim_{r \downarrow 0} G(r) = 0, \tag{3.16}$$

where

$$G(r) = \int_0^{2\pi} r^2 |g(r\cos\theta, r\sin\theta)| d\theta \in C(0, \infty).$$

Then $x_i g \in L^1_{\mathrm{loc}}(\mathbf{R}^2)$ for each $i = 1, 2$ and the equation (3.14) holds in $\mathscr{D}'(\mathbf{R}^2)$.

Proof. We fix $\chi(r) \in C^\infty([0,\infty))$ satisfying

$$\chi(r) = \begin{cases} 1, & (1 \le r) \\ 0, & (0 \le r \le \frac{1}{2}) \end{cases} \quad \text{and} \quad 0 \le \chi(r) \le 1 \quad \text{for every } r.$$

For every $\varphi \in \mathscr{D}(\mathbf{R}^2)$ and every ε satisfying $0 < \varepsilon \ll 1$, we set

$$\varphi_\varepsilon(x) := \varphi(x)\chi\left(\frac{|x|}{\varepsilon}\right) \in \mathscr{D}(\mathbf{R}\backslash\{0\}).$$

Then we get

$$\int_{\mathbf{R}^2} g(x)(x\cdot\nabla)\varphi_\varepsilon(x) = \int_{\mathbf{R}^2} f(x)\varphi_\varepsilon(x)$$

from the equation (3.14). Here we have

$$\int_{\mathbf{R}^2} f(x)\varphi_\varepsilon(x) \longrightarrow \int_{\mathbf{R}^2} f(x)\varphi(x) \quad \text{as } \varepsilon \downarrow 0$$

from the condition (3.15).

On the other hand, we set

$$\begin{aligned}\int_{\mathbf{R}^2} g(x)(x\cdot\nabla)\varphi_\varepsilon(x) &= \int_{\mathbf{R}^2} g(x)\left\{(x\cdot\nabla)\varphi(x)\right\}\chi\left(\frac{|x|}{\varepsilon}\right) \\ &\quad + \int_{\mathbf{R}^2} g(x)\varphi(x)(x\cdot\nabla)\chi\left(\frac{|x|}{\varepsilon}\right) \\ &=: I_1 + I_2.\end{aligned}$$

Since we assumed (3.16), we have

$$\int_{B_1(0)\backslash B_\varepsilon(0)} |x_i g| \le \int_\varepsilon^1 \int_0^{2\pi} r^2 |g(r\cos\theta, r\sin\theta)| d\theta dr = \int_\varepsilon^1 G(r)dr = O(1)$$

as $\varepsilon \downarrow 0$, that means $x_i g \in L^1_{\mathrm{loc}}(\mathbf{R}^2)$. Therefore

$$I_1 \longrightarrow \int_{\mathbf{R}^2} g(x)(x\cdot\nabla)\varphi(x)$$

as $\varepsilon \downarrow 0$.

On the other hand, we have

$$|I_2| \le \frac{1}{\varepsilon}\sup|\varphi(x)|\sup|\chi'(r)| \int_{B_\varepsilon(0)\backslash B_{\frac{\varepsilon}{2}}(0)} |x||g(x)|dx$$

and

$$\frac{1}{\varepsilon}\int_{B_\varepsilon(0)\backslash B_{\frac{\varepsilon}{2}}(0)}|x||g(x)|dx = \frac{1}{\varepsilon}\int_{\frac{\varepsilon}{2}}^{\varepsilon}\int_0^{2\pi} r^2|g(r\cos\theta, r\sin\theta)|d\theta dr$$
$$= \frac{1}{\varepsilon}\int_{\frac{\varepsilon}{2}}^{\varepsilon} G(r)dr \leq \frac{1}{2}\sup_{\frac{\varepsilon}{2}\leq r\leq\varepsilon}|G(r)| \longrightarrow 0$$

from (3.16). □

Proof of Lemma 3.3. Since $p > 2$, the function ζ is well-defined. It is easy to see (3.9) since $\mathrm{div}(xe^\zeta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2e^\zeta\right)$. The estimate (3.10) is also obtained by the elementary calculations.

Since $f \in L^1(\mathbf{R}^2)$ from the estimate (3.7), we get

$$\int_0^r tf(t\cos\theta, t\sin\theta)dt$$
$$= -\int_r^\infty tf(t\cos\theta, t\sin\theta)dt + \lim_{r'\longrightarrow 0}\int_{r'}^\infty tf(t\cos\theta, t\sin\theta)dt$$
$$= -\int_r^\infty tf(t\cos\theta, t\sin\theta)dt$$

for every $\theta \in [0, 2\pi)$ by the assumption (3.11). Therefore

$$|r^2e^\zeta| = \left|-\int_r^\infty tf(t\cos\theta, t\sin\theta)dt\right| = \left|\int_0^r tf(t\cos\theta, t\sin\theta)dt\right|$$
$$\leq \int_0^r t\left|f(t\cos\theta, t\sin\theta)\right|dt. \tag{3.17}$$

Now set $g(x) = e^{\zeta(x)}$. Then

$$G(r) := \int_0^{2\pi} r^2|g(r\cos\theta, r\sin\theta)|d\theta$$
$$\leq \int_0^{2\pi}\int_0^r t|f(t\cos\theta, t\sin\theta)|dtd\theta = \int_{B_r(0)}|f(x)|dx \longrightarrow 0$$

as $r \downarrow 0$ because $f \in L^1(\mathbf{R}^2)$.

Therefore we get $x_ie^\zeta \in L^1_{\mathrm{loc}}(\mathbf{R}^2)$ and the equation (3.13) holds by Proposition 3.4. Moreover since we have (3.17) and (3.7), we get

$$|r^2e^\zeta| \leq \int_0^r tCdt = \frac{C}{2}r^2,$$

that is, (3.12) holds. □

Proof of Proposition 2.4. We would like to apply Lemma 3.1 and Lemma 3.3 to

$$f = f_n(\widetilde{y}) := e^{\widetilde{u}_n(\widetilde{y})}\widetilde{v}_n(\widetilde{y})\xi\left(\frac{|\delta_n \widetilde{y}|}{\bar{R}}\right)$$
$$\left(\longrightarrow \boldsymbol{a}\cdot\nabla\left(-\frac{1}{4}e^U\right) + b\,\mathrm{div}\left(\frac{1}{2}\widetilde{y}e^U\right)\right)$$

From (3.1) we are able to assume the conditions (3.2) and (3.7) hold with $p = 4$ uniformly with respect to n. The problem is that the conditions (3.5) and (3.11) do not necessarily hold for these f_n's. Therefore we *symmetrize* them.

Case 1: When $\boldsymbol{a} = (a, 0) \neq \mathbf{0}$, that is, $a \neq 0$. We note that

$$\boldsymbol{a}\cdot\nabla\left(-\frac{1}{4}e^U\right) = \frac{64a\widetilde{y}_1}{(8+|\widetilde{y}|^2)^3} \quad \text{and} \quad \mathrm{div}\left(\frac{1}{2}\widetilde{y}e^U\right) = \frac{64(8-|\widetilde{y}|^2)}{(8+|\widetilde{y}|^2)^3}$$

are odd and even functions with respect to y_1, respectively. Therefore we divide $f_n(\widetilde{y})$ into odd and even parts with respect to $\widetilde{y}_1$:

$$\begin{aligned} f_n(\widetilde{y}) &= \frac{f_n(\widetilde{y}_1,\widetilde{y}_2) - f_n(-\widetilde{y}_1,\widetilde{y}_2)}{2} + \frac{f_n(\widetilde{y}_1,\widetilde{y}_2) + f_n(-\widetilde{y}_1,\widetilde{y}_2)}{2} \\ &=: f_n^o(\widetilde{y}) + f_n^e(\widetilde{y}). \end{aligned}$$

Then we see

$$f_n^o(\widetilde{y}) \longrightarrow \frac{64a\widetilde{y}_1}{(8+|\widetilde{y}|^2)^3}, \quad f_n^e(\widetilde{y}) \longrightarrow \frac{64(8-|\widetilde{y}|^2)}{(8+|\widetilde{y}|^2)^3} \tag{3.18}$$

locally uniformly from (1.7). Here we note that

$$\mathrm{supp}\, f_n^o, \quad \mathrm{supp}\, f_n^e \subset \mathrm{supp}\, f_n \Subset B_{\frac{\bar{R}}{\delta_n}}(0)\,. \tag{3.19}$$

Corresponding to the division of f_n into two parts, we also divide $\psi_n(x)$ into two parts:

$$\begin{aligned} \psi_n(x) &= \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} G(x, \delta_n\widetilde{y} + x_n) f_n(\widetilde{y})d\widetilde{y} \\ &= \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} G(x, \delta_n\widetilde{y} + x_n) f_n^o(\widetilde{y})d\widetilde{y} + \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} G(x, \delta_n\widetilde{y} + x_n) f_n^e(\widetilde{y})d\widetilde{y} \\ &=: I_n^o + I_n^e. \end{aligned}$$

We note that

$$|f_n^o|,\ |f_n^e| \leq \frac{C'}{(1+|\widetilde{y}_1|+|\widetilde{y}_2|)^4} \quad \text{in } \mathbf{R}^2 \tag{3.20}$$

form (3.1). Moreover, it holds that

$$\int_{-\infty}^{\infty} f_n^o(\widetilde{y}_1,\widetilde{y}_2)d\widetilde{y}_1 = 0.$$

Therefore all the assumptions in Lemma 3.1 holds for f_n^o with $p=4$ uniformly with respect to n and we get the estimate (3.6) for corresponding ω_n uniformly.

Here we also note that

$$\omega_n = -\int_{\frac{\widetilde{y}_1}{a}}^{\infty} f_n^o(at,\widetilde{y}_2)dt = \int_{-\infty}^{\frac{\widetilde{y}_1}{a}} f_n^o(at,\widetilde{y}_2)dt$$

and consequently

$$\operatorname{supp}\omega_n \subset B_{\frac{\bar{R}}{\delta_n}}(0) \tag{3.21}$$

since (3.19).

From (3.18) and (3.20), we get

$$f_n^o(\widetilde{y}_1,\widetilde{y}_2) \longrightarrow \frac{64a\widetilde{y}_1}{(8+|\widetilde{y}|^2)^3} =: f_\infty^o \quad \text{in } L^1(\mathbf{R}) \text{ for every fixed } \widetilde{y}_2 \in \mathbf{R}.$$

Here we note that the function ω determined from f_∞^o by Lemma 3.1 is $\omega_\infty = -\frac{1}{4}e^U$ and we have

$$\begin{aligned} |\omega_n(\widetilde{y})-\omega_\infty(\widetilde{y})| &\leq \int_{\frac{\widetilde{y}_1}{a}}^{\infty} |f_n^o(at,\widetilde{y}_2)-f_\infty^o(at,\widetilde{y}_2)|dt \\ &\leq \frac{1}{a}\|f_n^o(\cdot,\widetilde{y}_2)-f_\infty^o(\cdot,\widetilde{y}_2)\|_{L^1(\mathbf{R})} \longrightarrow 0 \end{aligned}$$

as $n \longrightarrow \infty$ for every $\widetilde{y} \in \mathbf{R}^2$.

Using these facts, we get

$$\begin{aligned}
I_n^o &= \int_{B_{\frac{\tilde{R}}{\delta_n}}(0)} G(x, \delta_n \widetilde{y} + x_n) f_n^o(\widetilde{y}) d\widetilde{y} = \int_{B_{\frac{\tilde{R}}{\delta_n}}(0)} G(x, \delta_n \widetilde{y} + x_n) a \frac{\partial}{\partial \widetilde{y}_1} \omega_n d\widetilde{y} \\
&= -a\delta_n \int_{B_{\frac{\tilde{R}}{\delta_n}}(0)} \frac{\partial}{\partial y_1} G(x, \delta_n \widetilde{y} + x_n) \omega_n(\widetilde{y}) d\widetilde{y} \\
&= -a\delta_n \frac{\partial}{\partial y_1} G(x, x_n) \int_{B_{\frac{\tilde{R}}{\delta_n}}(0)} \omega_n(\widetilde{y}) d\widetilde{y} \\
&\quad - a\delta_n \int_{B_{\frac{\tilde{R}}{\delta_n}}(0)} \left\{ \frac{\partial}{\partial y_1} G(x, \delta_n \widetilde{y} + x_n) - \frac{\partial}{\partial y_1} G(x, x_n) \right\} \omega_n(\widetilde{y}) d\widetilde{y}
\end{aligned}$$

Here, from (3.6) that holds for ω_n with $p = 4$ and C' independent of n, we get

$$\int_{B_{\frac{\tilde{R}}{\delta_n}}(0)} \omega_n(\widetilde{y}) d\widetilde{y} \longrightarrow \int_{\mathbf{R}^2} \omega_\infty(\widetilde{y}) d\widetilde{y} = -\frac{1}{4} \int_{\mathbf{R}^2} e^U = -2\pi.$$

from the Lebesgue dominated convergence theorem.

On the other hand, since $\delta_n \widetilde{y} + x_n \in B_{\tilde{R}}(x_n)$ and $\operatorname{supp} \omega_n \left(\frac{y - x_n}{\delta_n} \right) \Subset B_{\tilde{R}}(x_n)$, there exists a constant $C'' > 0$ independent of x and we have the estimate

$$\begin{aligned}
&\left| \int_{B_{\frac{\tilde{R}}{\delta_n}}(0)} \left\{ \frac{\partial}{\partial y_1} G(x, \delta_n \widetilde{y} + x_n) - \frac{\partial}{\partial y_1} G(x, x_n) \right\} \omega_n(\widetilde{y}) d\widetilde{y} \right| \\
&\leq C'' \delta_n \int_{B_{\frac{\tilde{R}}{\delta_n}}(0)} |\widetilde{y}| |\omega_n(\widetilde{y})| d\widetilde{y} = O(\delta_n) = o(1)
\end{aligned}$$

similarly from the Lebesgue dominated convergence theorem because of (3.6) for ω_n with $p = 4$.

Consequently we get

$$I_n^o = 2\pi \boldsymbol{a} \cdot \nabla_y G(x, x_n) \delta_n + o(\delta_n) \tag{3.22}$$

for every $x \in \overline{\Omega} \backslash B_{\tilde{R}}(x_n)$.

Next we calculate I_n^e part with Lemma 3.3. To this purpose we set

$$f_\infty^e := \operatorname{div} \left(\frac{1}{2} \widetilde{y} e^U \right) = \frac{64(8 - |\widetilde{y}|^2)}{(8 + |\widetilde{y}|^2)^3}.$$

and observe that

$$f_\infty^e\left(\frac{8\widetilde{y}}{|\widetilde{y}|^2}\right) = -\frac{|\widetilde{y}|^4}{64} f_\infty^e(\widetilde{y})$$

Corresponding to this symmetry of the Kelvin transformation, we divide f_n^e into two parts:

$$\begin{aligned} f_n^e(\widetilde{y}) &= \frac{1}{2}\left\{f_n^e(\widetilde{y}) - \frac{64}{|\widetilde{y}|^4} f_n^e\left(\frac{8\widetilde{y}}{|\widetilde{y}|^2}\right)\right\} + \frac{1}{2}\left\{f_n^e(\widetilde{y}) + \frac{64}{|\widetilde{y}|^4} f_n^e\left(\frac{8\widetilde{y}}{|\widetilde{y}|^2}\right)\right\} \\ &=: f_n^{e,-}(\widetilde{y}) + f_n^{e,+}(\widetilde{y}) \quad \text{for } \widetilde{y} \in \mathbf{R}^2\backslash\{0\}. \end{aligned}$$

Then it holds that

$$f_n^{e,-}\left(\frac{8\widetilde{y}}{|\widetilde{y}|^2}\right) = -\frac{|\widetilde{y}|^4}{64} f_n^{e,-}(\widetilde{y}) \quad \text{and} \quad f_n^{e,+}\left(\frac{8\widetilde{y}}{|\widetilde{y}|^2}\right) = \frac{|\widetilde{y}|^4}{64} f_n^{e,+}(\widetilde{y}).$$

Moreover we have

$$f_n^{e,-}(\widetilde{y}) \longrightarrow f_\infty^e(\widetilde{y}) \quad \text{locally uniformly in } \mathbf{R}^2\backslash\{0\}, \tag{3.23}$$

$$f_n^{e,+}(\widetilde{y}) \longrightarrow 0 \quad \text{locally uniformly in } \mathbf{R}^2\backslash\{0\} \tag{3.24}$$

from (3.18) and

$$|f_n^{e,-}|,\ |f_n^{e,+}| \le \frac{C'}{(1+|\widetilde{y}|)^4} \quad \text{in } \mathbf{R}^2\backslash\{0\} \tag{3.25}$$

for some constant C' independent of n from (3.20). Especially (3.7) holds for $f = f_n^{e,-}$ and $p = 4$.

Now we note that

$$\begin{aligned} &\lim_{r\longrightarrow 0}\int_r^\infty t f_n^{e,-}(t\cos\theta, t\sin\theta)dt \\ &= \lim_{r\longrightarrow 0}\int_r^\infty \frac{t}{2}\left\{f_n^e(t\cos\theta, t\sin\theta) - \frac{64}{t^4} f_n^e\left(\frac{8\cos\theta}{t}, \frac{8\sin\theta}{t}\right)\right\} dt \end{aligned}$$

for each $\theta \in [0, 2\pi)$. Here

$$\int_r^\infty t f_n^e(t\cos\theta, t\sin\theta)dt \longrightarrow \int_0^\infty t f_n^e(t\cos\theta, t\sin\theta)dt$$

since f_n^e is continuous at 0 and

$$\begin{aligned} \int_r^\infty \frac{64}{t^3} f_n^e\left(\frac{8\cos\theta}{t}, \frac{8\sin\theta}{t}\right) dt &= \int_0^{\frac{8}{r}} t f_n^e\left(t\cos\theta, t\sin\theta\right) dt \\ &\longrightarrow \int_0^\infty t f_n^e\left(t\cos\theta, t\sin\theta\right) dt \end{aligned}$$

as $r \downarrow 0$ from (3.20). Therefore we get (3.11) for $f = f_n^{e,-}$ and consequently all the conclusions in Lemma 3.3.

Taking another cut-off function $\eta(r) \in C_0^\infty([0,\infty))$ satisfying $\operatorname{supp}\eta \subset [0,1)$ and $\eta \equiv 1$ on $\operatorname{supp}\xi$. Then we get

$$
\begin{aligned}
I_n^e &= \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} G(x, \delta_n\widetilde{y} + x_n) f_n^e(\widetilde{y}) d\widetilde{y} \\
&= \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} G(x, \delta_n\widetilde{y} + x_n)\eta\left(\frac{\delta_n|\widetilde{y}|}{\bar{R}}\right) f_n^e(\widetilde{y}) d\widetilde{y} \\
&= \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} G(x, \delta_n\widetilde{y} + x_n)\eta\left(\frac{\delta_n|\widetilde{y}|}{\bar{R}}\right) \left\{f_n^{e,-}(\widetilde{y}) + f_n^{e,+}(\widetilde{y})\right\} d\widetilde{y} \\
&=: I_n^{e,-} + I_n^{e,+}.
\end{aligned}
$$

Since $x \in \overline{\Omega}\backslash B_{\bar{R}}(x_n)$, we may assume

$$
G(x, \delta_n\widetilde{y} + x_n)\eta\left(\frac{\delta_n|\widetilde{y}|}{\bar{R}}\right) \in \mathscr{D}'(\mathbf{R}^2)
$$

for each fixed x. Let ζ_n be the function ζ determined from $f = f_n^{e,-}$ in Lemma 3.3. Then we have

$$
\begin{aligned}
I_n^{e,-} = &-\delta_n \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} \widetilde{y}\cdot\nabla_y G(x, \delta_n\widetilde{y} + x_n)\eta\left(\frac{\delta_n|\widetilde{y}|}{\bar{R}}\right) e^{\zeta_n} d\widetilde{y} \\
&- \frac{\delta_n}{\bar{R}} \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} |\widetilde{y}| G(x, \delta_n\widetilde{y} + x_n)\eta'\left(\frac{\delta_n|\widetilde{y}|}{\bar{R}}\right) e^{\zeta_n} d\widetilde{y}
\end{aligned}
\tag{3.26}
$$

Immediately (3.10) and (3.12) give

$$
\frac{1}{\bar{R}} \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} |\widetilde{y}| G(x, \delta_n\widetilde{y} + x_n)\eta'\left(\frac{\delta_n|\widetilde{y}|}{\bar{R}}\right) e^{\zeta_n} d\widetilde{y} = o(1)
$$

as $n \longrightarrow \infty$ since $\eta'\left(\frac{\delta_n|\widetilde{y}|}{\bar{R}}\right) \longrightarrow 0$ for every $\widetilde{y} \in \mathbf{R}^2$ and the Lebesgue convergence theorem.

We note $\zeta_\infty := \log \frac{32}{(8+|x|^2)^2}$ is the corresponding ζ to $f_\infty^{e,-}$ and it hold

$$
|r^2 e^{\zeta_n} - r^2 e^{\zeta_\infty}| \le \int_r^\infty t|f_n^{e,-} - f_\infty^{e,-}| dt \longrightarrow 0
$$

from (3.23), (3.25), and the Lebesgue convergence theorem. Especially we have

$$e^{\zeta_n(\widetilde{y})} \longrightarrow e^{\zeta_\infty(\widetilde{y})} \quad \text{for every } \widetilde{y} \in \mathbf{R}^2 \backslash \{0\}. \tag{3.27}$$

Applying these facts (3.27), (3.10), and (3.12) to (3.26), we get

$$I_n^{e,-} = -\delta_n \left(\nabla_y G(x,0) \cdot \int_{\mathbf{R}^2} \widetilde{y} e^{\zeta_\infty(\widetilde{y})} d\widetilde{y} + o(1) \right) + o(\delta_n) = o(\delta_n) \tag{3.28}$$

as $n \longrightarrow \infty$ since $\int_{\mathbf{R}^2} \widetilde{y} e^{\zeta_\infty(\widetilde{y})} = \int_{\mathbf{R}^2} \frac{32\widetilde{y}}{(8+|\widetilde{y}|^2)^2} = 0$.

Finally we calculate $I_n^{e,+}$. We are able to divide $I_n^{e,+}$ into two more parts:

$$\begin{aligned} I_n^{e,+} &= \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} G(x, \delta_n \widetilde{y} + x_n) \eta \left(\frac{\delta_n |\widetilde{y}|}{\bar{R}} \right) f_n^{e,+}(\widetilde{y}) d\widetilde{y} \\ &= G(x, x_n) \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} \eta \left(\frac{\delta_n |\widetilde{y}|}{\bar{R}} \right) f_n^{e,+}(\widetilde{y}) d\widetilde{y} \\ &\quad + \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} \{ G(x, \delta_n \widetilde{y} + x_n) - G(x, x_n) \} \eta \left(\frac{\delta_n |\widetilde{y}|}{2\bar{R}} \right) f_n^{e,+}(\widetilde{y}) d\widetilde{y}. \end{aligned}$$

Here

$$\begin{aligned} &\left| \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} \{ G(x, \delta_n \widetilde{y} + x_n) - G(x, x_n) \} \eta \left(\frac{\delta_n |\widetilde{y}|}{\bar{R}} \right) f_n^{e,+}(\widetilde{y}) d\widetilde{y} \right| \\ &\leq C''' \delta_n \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} |\widetilde{y}| \eta \left(\frac{\delta_n |\widetilde{y}|}{\bar{R}} \right) \left| f_n^{e,+}(\widetilde{y}) \right| d\widetilde{y} = o(\delta_n) \end{aligned}$$

for some constant $C''' > 0$ because $x \in \overline{\Omega} \backslash B_{\bar{R}}(x_n)$, $\operatorname{supp} \eta \left(\frac{|y - x_n|}{\bar{R}} \right) \Subset B_{\bar{R}}(x_n)$, (3.24), (3.25), and the Lebesgue convergence theorem.

On the other hand, we have

$$
\begin{aligned}
&\int_{B_{\frac{\bar{R}}{\delta_n}}(0)} \eta\left(\frac{\delta_n|\widetilde{y}|}{\bar{R}}\right) f_n^{e,+}(\widetilde{y})d\widetilde{y} \\
&= \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} \xi\left(\frac{\delta_n|\widetilde{y}|}{\bar{R}}\right)\cdot\frac{1}{2}\left\{f_n^e(\widetilde{y}) + \frac{64}{|\widetilde{y}|^4} f_n^e\left(\frac{8\widetilde{y}}{|\widetilde{y}|^2}\right)\right\} d\widetilde{y} \\
&= \frac{1}{2}\int_{B_{\frac{\bar{R}}{\delta_n}}(0)} \eta\left(\frac{\delta_n|\widetilde{y}|}{\bar{R}}\right) f_n^e(\widetilde{y})d\widetilde{y} + \frac{1}{2}\int_{\mathbf{R}^2\setminus B_{\frac{8\delta_n}{\bar{R}}}(0)} \eta\left(\frac{\delta_n}{\bar{R}}\cdot\frac{8\widetilde{y}}{|\widetilde{y}|^2}\right) f_n^e(\widetilde{y})d\widetilde{y} \\
&= \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} \eta\left(\frac{\delta_n|\widetilde{y}|}{\bar{R}}\right) f_n^e(\widetilde{y})d\widetilde{y} \\
&\quad + \frac{1}{2}\int_{B_{\frac{\bar{R}}{\delta_n}}(0)} \left\{\eta\left(\frac{\delta_n}{\bar{R}}\cdot\frac{8\widetilde{y}}{|\widetilde{y}|^2}\right) - \eta\left(\frac{\delta_n|\widetilde{y}|}{\bar{R}}\right)\right\} f_n^e(\widetilde{y})d\widetilde{y}.
\end{aligned}
$$

Here we recall that $\eta\left(\frac{\delta_n|\widetilde{y}|}{\bar{R}}\right) \equiv 1$ on $\operatorname{supp} f_n^e \subset B_{\frac{\bar{R}}{\delta_n}}(0)$. Therefore

$$
\int_{B_{\frac{\bar{R}}{\delta_n}}(0)} \eta\left(\frac{\delta_n|\widetilde{y}|}{\bar{R}}\right) f_n^e(\widetilde{y})d\widetilde{y} = \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} f_n^e(\widetilde{y})d\widetilde{y} = \int_{B_{\frac{\bar{R}}{\delta_n}}(0)} f_n(\widetilde{y})d\widetilde{y} = \gamma_n.
$$

Here we recall that $\eta \equiv 1$ on $\operatorname{supp}\xi$. Therefore we see

$$
\operatorname{supp}\left\{\eta\left(\frac{\delta_n}{\bar{R}}\cdot\frac{8\widetilde{y}}{|\widetilde{y}|^2}\right) - \eta\left(\frac{\delta_n|\widetilde{y}|}{\bar{R}}\right)\right\} f_n^e(\widetilde{y}) \subset B_{\frac{8\delta_n}{\widetilde{R}}}(0),
$$

where $\widetilde{R} > 0$ is any fixed number satisfying $[0, \widetilde{R}) \subset \operatorname{supp}\xi$. We note that $f_n^e(\widetilde{y}) = O(1)$ on $B_{\frac{8\delta_n}{\widetilde{R}}}(0)$ and consequently we get

$$
\int_{B_{\frac{\bar{R}}{\delta_n}}(0)} \left\{\eta\left(\frac{\delta_n}{\bar{R}}\cdot\frac{8\widetilde{y}}{|\widetilde{y}|^2}\right) - \eta\left(\frac{\delta_n|\widetilde{y}|}{\bar{R}}\right)\right\} f_n^e(\widetilde{y})d\widetilde{y} = O(\delta_n^2) = o(\delta_n).
$$

Summarizing these we get

$$
I_n^{e,+} = G(x, x_n)\gamma_n + o(\delta_n)
$$

for every $x \in \overline{\Omega}\setminus B_{\bar{R}}(x_n)$ and the conclusion.

Case 2: When $\boldsymbol{a} = (a, 0) = \mathbf{0}$, that is, $a = 0$. In this case we may apply the analysis for f_n^e in Case 1 to f_n. Indeed, we use the property of even functions with respect to $\widetilde{y}_1$ only to determine the limit function in (3.23) and (3.24). To get similar result, we divide f_n into following two parts here:

$$\begin{aligned} f_n(\widetilde{y}) &= \frac{1}{2}\left\{ f_n(\widetilde{y}) - \frac{64}{|\widetilde{y}|^4} f_n\left(\frac{8\widetilde{y}}{|\widetilde{y}|^2}\right)\right\} + \frac{1}{2}\left\{ f_n(\widetilde{y}) + \frac{64}{|\widetilde{y}|^4} f_n\left(\frac{8\widetilde{y}}{|\widetilde{y}|^2}\right)\right\} \\ &=: f_n^-(\widetilde{y}) + f_n^+(\widetilde{y}) \quad \text{for } \widetilde{y} \in \mathbf{R}^2\backslash\{0\}. \end{aligned}$$

We note that the function $f_\infty^o(\widetilde{y}) = \frac{64 a \widetilde{y}_1}{(8+|\widetilde{y}|^2)^3}$ satisfies

$$\frac{64}{|\widetilde{y}|^4} f_\infty^o\left(\frac{8\widetilde{y}}{|\widetilde{y}|^2}\right) = f_\infty^o(\widetilde{y})$$

for every $a \geq 0$, that is,

$$f_n^+ \longrightarrow f_\infty^o$$

locally uniformly in general from (1.7). Since $a = 0$ now, we have following behaviors instead of (3.23) and (3.24):

$$\begin{aligned} f_n^-(\widetilde{y}) &\longrightarrow f_\infty^e(\widetilde{y}) \quad \text{locally uniformly in } \mathbf{R}^2\backslash\{0\}, \\ f_n^+(\widetilde{y}) &\longrightarrow f_\infty^o \equiv 0 \quad \text{locally uniformly in } \mathbf{R}^2\backslash\{0\}. \end{aligned}$$

It also holds that

$$|f_n^-|,\ |f_n^+| \leq \frac{C'}{(1 + |\widetilde{y}|)^4} \quad \text{in } \mathbf{R}^2\backslash\{0\}$$

for some constant C' independent of n from (3.1) instead of (3.25). Therefore we get the conclusion similarly to Case 1. □

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